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# Analytic Solutions of Teukolsky Equation in Kerr-de Sitter and Kerr-Newman-de Sitter Geometries

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## Abstract

The analytic solution of Teukolsky equation in Kerr-de Sitter and Kerr-Newman-de Sitter geometries is presented and the properties of the solution are examined. In particular, we show that our solution satisfies the Teukolsky-Starobinsky identities explicitly and fix the relative normalization between solutions with the spin weight  $s$  and  $-s$ .

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# 1 Introduction

In our series of papers, Mano, Suzuki and Takasugi [1, 2] have constructed the analytic solution of the perturbation equation of massless fields in Kerr geometries which is commonly called Teukolsky equation [3]. The solution enabled us to investigate various properties of black hole analytically [4], the scattering problem of particle emitted in black hole, the analytical expression for the absorption rate by Kerr black hole which was one of the main concern by Chandrasekhar [5], and the Teukolsky-Starobinsky identities.

Our solution is expressed in the form of series of hypergeometric functions and Coulomb wave functions. The coefficients of these series are determined by solving three term recurrence relations. Two series have different convergence regions and they are matched in the region where both series converge. This method turns out to be quite powerful in a practical calculation and is successfully applied to examine gravitational waves from a binary in a post-Newtonian expansion; to construct the template of the gravitational wave emitted from a particle moving around a Kerr black hole [6] and the rate of the gravitational wave absorbed by the black hole [7].

In our previous paper [8], Suzuki, Takasugi and Umetsu have extended this method to solve the perturbation equations of massless fields (the Teukolsky equation) in Kerr-de Sitter and Kerr-Newman-de Sitter geometries. We found the transformations such that both the angular and the radial equations are reduced to the Heun's equation [9]. The solution of Heun's equation is expressed in the form of series of hypergeometric functions and its coefficients are determined by three term recurrence relations, similarly to the Kerr geometry case. It should be noted that electromagnetic field and gravitational fields couple each other in Kerr-Newman-de Sitter geometries and do not follow the equation we considered there, although the fields follow the equation in Kerr-de Sitter geometries.

The solution of the radial Teukolsky equation given in Ref.[8] is convergent for  $r < r'_+$  with  $r'_+$  being the de Sitter horizon. In this paper, we give the solution valid in the entire physical region. This is achieved by constructing another solution which is convergent for  $r_+ < r$  and then by matching this solution and the previous solution in the region where both solutions are convergent. Then, we examine properties of the solution in detail. In particular, we show analytically that our solution satisfies the Teukolsky-Starobinsky (T-S) identities [10], if we properly take the relative normalization between the solution with the spin weight  $s$  and the one with  $-s$ .

It is quite surprising that the analytic solution is obtained for Kerr-de Sitter and Kerr-

Newman-de Sitter geometries, similarly to the Kerr geometry case. This might be the reflection of an underlying symmetry which the gravitational theory contains. Although the use of this solution is less clear in comparison with that in the Kerr geometries, we believe that our finding is important not only because the technique we used can be extended to other phenomena, but also because the solution may become relevant to the early stage of universe where the cosmological constant play important role.

In Sec.2, we review the transformation of the radial equation to Heun's equation, because we change some definitions of parameters from the previous paper[8]. In Sec.3, the solution is given in the form of series of hypergeometric functions. In particular, we present the solution which is convergent around the de Sitter horizon. We discuss the convergence region of the solution. Then, we match two solutions in the region where both solutions converge. In Sec.4, we explicitly show that our solution satisfies the Teukolsky-Starobinsky identities and the relative normalization between the solutions with the spin weight  $s$  and  $-s$ . We found some identities involving coefficients of series. Summary and discussions are given in Sec.5.

## 2 Teukolsky equation for the Kerr(-Newman)-de Sitter geometry and Heun's equation

We consider the Teukolsky equations for the Kerr-Newman-de Sitter geometries. In the Boyer-Lindquist coordinates the Kerr-Newman-de Sitter metric has the form,

$$ds^2 = -\rho^2 \left( \frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} \right) - \frac{\Delta_\theta \sin^2 \theta}{(1+\alpha)^2 \rho^2} [adt - (r^2 + a^2)d\varphi]^2 + \frac{\Delta_r}{(1+\alpha)^2 \rho^2} (dt - a \sin^2 \theta d\varphi)^2, \quad (2.1)$$

where

$$\begin{aligned} \Delta_r &= (r^2 + a^2) \left( 1 - \frac{\alpha}{a^2} r^2 \right) - 2Mr + Q^2 = -\frac{\alpha}{a^2} (r - r_+)(r - r_-)(r - r'_+)(r - r'_-), \\ \Delta_\theta &= 1 + \alpha \cos^2 \theta, & \alpha &= \frac{\Lambda a^2}{3}, \\ \bar{\rho} &= r + ia \cos \theta, & \rho^2 &= \bar{\rho} \bar{\rho}^*. \end{aligned} \quad (2.2)$$

Here  $\Lambda$  is the cosmological constant,  $M$  is the mass of the black hole,  $aM$  its angular momentum and  $Q$  its charge.

We assume that the time and the azimuthal angle dependence of the field has the form  $e^{-i(\omega t - m\phi)}$ . Then, the equation for the radial part with spin  $s$  and charge  $e$  is given by

$$\left[ \Delta_r \mathcal{D}_1 \mathcal{D}_s^\dagger + 2(1 + \alpha)(2s - 1)i\omega r - \frac{2\alpha}{a^2}(s - 1)(2s - 1) \right. \\ \left. + \frac{-2(1 + \alpha)eQKr + iseQr\partial_r \Delta_r + e^2 Q^2 r^2}{\Delta_r} - 2iseQ - s(1 - \alpha) - \lambda_s \right] R_s = 0, \quad (2.3)$$

where

$$\begin{aligned} \mathcal{D}_n &= \partial_r - \frac{i(1 + \alpha)K}{\Delta_r} + n \frac{\partial_r \Delta_r}{\Delta_r}, \\ \mathcal{D}_n^\dagger &= \partial_r + \frac{i(1 + \alpha)K}{\Delta_r} + n \frac{\partial_r \Delta_r}{\Delta_r}, \end{aligned} \quad (2.4)$$

with  $K = \omega(r^2 + a^2) - am$ . In our previous paper [8], we showed that solutions are characterized by the characteristic exponent (shifted angular momentum)  $\nu$ .

In this paper, we use slightly different definitions of the separation constant  $\lambda_s$  and the characteristic exponents  $\nu$  and  $-\nu - \omega$ , where  $\omega$  is the parameter appearing in the Heun's equation, is given explicitly in Eq.(2.17) and is different from the angular frequency in Eq.(2.3). We redefined  $\lambda_s$  and  $\nu$  as

$$\begin{aligned} \lambda_s &\rightarrow \lambda_s + s(1 - \alpha), \\ \nu &\rightarrow \nu - \frac{\omega - 1}{2}. \end{aligned} \quad (2.5)$$

With these modifications, as we showed in Ref.[8],  $\lambda_s$  becomes an even function of spin variable  $s$ ,

$$\lambda_s = \lambda_{-s}, \quad (2.6)$$

and if  $\nu$  is a characteristic exponent, then  $-\nu - 1$  becomes the one. That is, both

$$\nu \quad \text{and} \quad -\nu - 1. \quad (2.7)$$

become characteristic exponents.

With a new  $\lambda_s$ , the radial Teukolsky equation is explicitly written by

$$\left\{ \Delta_r^{-s} \frac{d}{dr} \Delta_r^{s+1} \frac{d}{dr} + \frac{1}{\Delta_r} \left[ (1 + \alpha)^2 \left( K - \frac{eQr}{1 + \alpha} \right)^2 - is(1 + \alpha) \left( K - \frac{eQr}{1 + \alpha} \right) \frac{d\Delta_r}{dr} \right] \right. \\ \left. + 4is(1 + \alpha)\omega r - \frac{2\alpha}{a^2}(s + 1)(2s + 1)r^2 + s(1 - \alpha) - 2iseQ - \lambda_s \right\} R_s = 0. \quad (2.8)$$

This equation has five regular singularities at  $r_{\pm}, r'_{\pm}$  and  $\infty$  which are assigned such that  $r_{\pm} \rightarrow M \pm \sqrt{M^2 - a^2 - Q^2}$  and  $r'_{\pm} \rightarrow \pm \frac{a}{\sqrt{\alpha}}$  in the limit  $\alpha \rightarrow 0$  ( $\Lambda \rightarrow 0$ ).

Next, we use a variable  $x$  rather than  $z$  used in our previous paper [8], which are defined by

$$x = 1 - z = \frac{(r_- - r'_-)(r - r_+)}{(r_- - r_+)(r - r'_-)} . \quad (2.9)$$

This transformation from  $r$  to  $x$  maps the inner horizon  $r_-$ , the outer horizon  $r_+$ , the de Sitter horizon  $r'_+$ ,  $r'_-$  and  $\infty$  to 0, 1,  $x_r$ ,  $\infty$  and  $x_{\infty}$ , respectively,

$$\begin{aligned} x_r &= 1 - z_r = \frac{(r_- - r'_-)(r'_+ - r_+)}{(r_- - r_+)(r'_+ - r'_-)} , \\ x_{\infty} &= 1 - z_{\infty} = \frac{(r_- - r'_-)}{(r_- - r_+)} . \end{aligned} \quad (2.10)$$

In the followings, we omit the subscript  $s$  for the radial wave function.

To proceed further, we define the following parameters,

$$A_{i\pm} = \frac{1}{2} \{-s \pm (2a_i + s)\} \quad (i = 1, 2, 3, 4), \quad (2.11)$$

Here  $a$ 's are pure imaginary numbers defined by

$$\begin{aligned} a_1 &= i \frac{a^2(1+\alpha)}{\alpha} \frac{(\omega(r_+^2 + a^2) - am - \frac{eQr_+}{1+\alpha})}{(r'_+ - r_+)(r'_- - r_+)(r_- - r_+)} , \\ a_2 &= i \frac{a^2(1+\alpha)}{\alpha} \frac{(\omega(r_-^2 + a^2) - am - \frac{eQr_-}{1+\alpha})}{(r'_+ - r_-)(r'_- - r_-)(r_+ - r_-)} , \\ a_3 &= i \frac{a^2(1+\alpha)}{\alpha} \frac{(\omega(r_+'^2 + a^2) - am - \frac{eQr'_+}{1+\alpha})}{(r_- - r'_+)(r'_- - r'_+)(r_+ - r'_+)} , \\ a_4 &= i \frac{a^2(1+\alpha)}{\alpha} \frac{(\omega(r_-'^2 + a^2) - am - \frac{eQr'_-}{1+\alpha})}{(r_- - r'_-)(r'_+ - r'_-)(r_+ - r'_-)} \end{aligned} \quad (2.12)$$

and they satisfy the relation

$$a_1 + a_2 + a_3 + a_4 = 0. \quad (2.13)$$

Now, following the transformation given in Ref.[8], we can factor out the singularity at  $x_{\infty}$  by the transformation as

$$R_{in,\{0,1\};s}^{\nu}(x) = (-x)^{A_1}(1-x)^{A_2} \left( \frac{x-x_r}{1-x_r} \right)^{A_3} \left( \frac{x-x_{\infty}}{1-x_{\infty}} \right)^{2s+1} f_{\{0,1\};s}^{\nu}(x) , \quad (2.14)$$

where we choose  $A_1 = A_{1-}$  for the solution to satisfy the incoming boundary condition at the outer horizon, and the other  $A_i (i = 2, 3)$  takes either  $A_{i+}$  or  $A_{i-}$ . Then, we find that  $f_{\{0,1\}}^\nu(x)$  satisfies the following Heun's equation

$$\left\{ \frac{d^2}{dx^2} + \left[ \frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-x_r} \right] \frac{d}{dx} + \frac{\sigma_+ \sigma_- x + v}{x(x-1)(x-x_r)} \right\} f_{\{0,1\};s}^\nu(x) = 0, \quad (2.15)$$

where

$$\begin{aligned} \gamma &= 2A_1 + s + 1, & \delta &= 2A_2 + s + 1, & \epsilon &= 2A_3 + s + 1, \\ \sigma_\pm &= A_1 + A_2 + A_3 + A_{4\mp} + 2s + 1. \end{aligned} \quad (2.16)$$

Parameters  $\gamma$ ,  $\delta$ ,  $\epsilon$  and  $\sigma_\pm$  satisfy the following relation,

$$\gamma + \delta - 1 = \sigma_+ + \sigma_- - \epsilon \equiv \omega, \quad (2.17)$$

which is required for Eq.(2.15) to be a Heun's equation. It should be noted that the parameter  $\omega$  defined above is a commonly used notation for Heun's equation and is different from the angular frequency. The remaining parameter  $v$  is given by

$$\begin{aligned} v &= -\frac{2a^4(1+\alpha)^2(r_- - r'_+)^2(r_- - r'_-)^2(r_+ - r'_-)(r'_+ - r'_-)}{\alpha^2 \mathcal{D}} \\ &\quad \left\{ -\omega^2 r_+^3(r_+ r_- - 2r_- r'_+ + r_+ r'_+) + 2a\omega(a\omega - m)r_+(r_- r'_+ - r_+^2) \right. \\ &\quad - a^2(a\omega - m)^2(2r_+ - r_- - r'_+) + \frac{eQ}{1+\alpha}[\omega r_+^2(r_+ r_- + r_+^2 - 3r_- r'_+ + r_+ r'_+) \\ &\quad \left. - a(a\omega - m)(r_+ r_- - 3r_+^2 + r_- r'_+ + r_+ r'_+)] + \left(\frac{eQ}{1+\alpha}\right)^2 r_+(-r_+^2 + r_- r'_+) \right\} \\ &\quad - \frac{2isa^2(1+\alpha)}{\alpha} \left[ \frac{\omega(r_+ r'_- + a^2) - am - \frac{eQ}{1+\alpha} \frac{r_+ + r'_-}{2}}{(r_+ - r_-)(r'_+ - r'_-)(r_+ - r'_-)} \right] \\ &\quad - (s+1) \left( (2s+1) \left[ \frac{2r_-'^2}{(r_+ - r_-)(r'_+ - r'_-)} + x_\infty \right] + [(1+x_r)A_1 + x_r A_2 + A_3] \right) \\ &\quad - 2A_1(x_r A_2 + A_3) + \frac{a^2}{\alpha(r_+ - r_-)(r'_+ - r'_-)} [-\lambda_s + s(1-\alpha)], \end{aligned} \quad (2.18)$$

where  $\mathcal{D}$  is the discriminant of  $\Delta_r = 0$ ,

$$\begin{aligned} \mathcal{D} &= (r_+ - r_-)^2(r_+ - r'_+)^2(r_+ - r'_-)^2(r_- - r'_+)^2(r_- - r'_-)^2(r'_+ - r'_-)^2 \\ &= \frac{16a^{10}}{\alpha^5} \left\{ (1-\alpha)^3 [M^2 - (1-\alpha)(a^2 + Q^2)] - \frac{16\alpha^2}{a^4} (a^2 + Q^2)^3 \right. \\ &\quad \left. + \frac{\alpha}{a^2} [-27M^4 + 36(1-\alpha)M^2(a^2 + Q^2) - 8(1-\alpha)^2(a^2 + Q^2)^2] \right\}. \end{aligned} \quad (2.19)$$

A parameter  $\omega$  in the definition of  $v$  is an angular frequency.

Modifications in Eq.(2.5) and the use of the variable  $x$  rather than  $z$  cause changes of parameters in the Heun's equation from those in Ref.[8] . The changes of parameters are shown below:

$$\begin{aligned} B_1 &\rightarrow A_2, \quad B_2 \rightarrow A_1, \quad B_3 \rightarrow A_3, \\ (\alpha, \beta) &\rightarrow (\sigma_+, \sigma_-), \quad -q \rightarrow v, \quad v \rightarrow -v - \sigma_+ \sigma_-, \quad a_H \rightarrow x_r. \end{aligned} \quad (2.20)$$

Parameters in the left-hand side are those used in Ref.[8] and those in the right-hand side are those used in this paper. The  $v$  in Eq.(2.18) is derived from the corresponding parameter  $v$  of Eq.(3.10) in Ref.[8] by exchanging  $r_+$  and  $r_-$  and changing of  $\lambda_s$  in Eq.(2.5). There was a miss-print in  $v$  in our previous paper [8] where  $-\lambda_s - 2iseQ$  should be read  $-\lambda_s$ .

### 3 Solutions of Teukolsky equation

In this section, we derive the solutions which are convergent in the region including the outer and the inner horizon and also the solutions which are convergent in the region including the de Sitter horizon and  $\infty$ . Then, by matching these solutions in the region where both solutions are convergent, we obtain the solutions valid in the entire region.

#### (A) Solutions convergent around outer horizon

There are two independent solutions; one satisfies the incoming boundary condition at the outer horizon and the other does the outgoing boundary condition. They are expressed as series of hypergeometric functions and these series converges in the region,  $r < r'_+$  with  $r'_+$  being the de Sitter horizon.

*(a-1) The solution which satisfies the incoming boundary condition at the outer horizon*

The solution is given by taking into account of changes in Eq.(2.5) and adopting the variable  $x$  as

$$\begin{aligned} R_{in;\{0,1\};s}^\nu(x) &= (-x)^{A_1} (1-x)^{A_2} \left( \frac{x-x_r}{1-x_r} \right)^{A_3} \left( \frac{x-x_\infty}{1-x_\infty} \right)^{2s+1} f_{\{0,1\};s}^\nu(x), \\ f_{\{0,1\};s}^\nu(x) &= \sum_{n=-\infty}^{\infty} a_n^\nu F \left( -n - \nu + \frac{\omega}{2} - \frac{1}{2}, n + \nu + \frac{\omega}{2} + \frac{1}{2}; \gamma; x \right), \end{aligned} \quad (3.1)$$

where  $\nu$  is the characteristic exponent (the shifted angular momentum). Coefficients  $a_n^\nu$  are determined by solving the following three term recurrence relation with the initial

condition  $a_0^\nu = 1$ ,

$$\alpha_n^\nu a_{n+1}^\nu + \beta_n^\nu a_n^\nu + \gamma_n^\nu a_{n-1}^\nu = 0, \quad (3.2)$$

where

$$\begin{aligned} \alpha_n^\nu &= -\frac{(n + \nu - \frac{\omega}{2} + \frac{3}{2})(n + \nu - \sigma_+ + \frac{\omega}{2} + \frac{3}{2})(n + \nu - \sigma_- + \frac{\omega}{2} + \frac{3}{2})(n + \nu + \delta - \frac{\omega}{2} + \frac{1}{2})}{2(n + \nu + 1)(2n + 2\nu + 3)}, \\ \beta_n^\nu &= \frac{(1 - \omega)(\gamma - \delta)(\sigma_+ - \sigma_- + \epsilon - 1)(\sigma_+ - \sigma_- - \epsilon + 1)}{32(n + \nu)(n + \nu + 1)} + \left(\frac{1}{2} - x_r\right)(n + \nu)(n + \nu + 1) \\ &\quad + \frac{1}{4}[\epsilon(\gamma - \delta) + \delta(1 - \omega) + 2\sigma_+\sigma_-] + \frac{\omega^2 - 1}{4}x_r + v, \\ \gamma_n^\nu &= -\frac{(n + \nu + \sigma_+ - \frac{\omega}{2} - \frac{1}{2})(n + \nu + \sigma_- - \frac{\omega}{2} - \frac{1}{2})(n + \nu + \gamma - \frac{\omega}{2} - \frac{1}{2})(n + \nu + \frac{\omega}{2} - \frac{1}{2})}{2(n + \nu)(2n + 2\nu - 1)}. \end{aligned} \quad (3.3)$$

*(a-2) Determination of the characteristic exponent*

The characteristic exponent  $\nu$  is determined such that the series of hypergeometric functions converges. Let us define the continued fractions

$$R_n(\nu) = \frac{a_n^\nu}{a_{n-1}^\nu}, \quad L_n(\nu) = \frac{a_n^\nu}{a_{n+1}^\nu}, \quad (3.4)$$

which satisfy

$$R_n(\nu) = -\frac{\gamma_n^\nu}{\beta_n^\nu + \alpha_n^\nu R_{n+1}(\nu)}, \quad L_n(\nu) = -\frac{\alpha_n^\nu}{\beta_n^\nu + \gamma_n^\nu L_{n-1}(\nu)}. \quad (3.5)$$

Now we can evaluate the coefficients  $a_n^\nu$  by using either series  $R_n(\nu)$  or  $L_n(\nu)$  with an appropriate initial data. The convergence of the series requires the following transcendental equation,

$$R_n(\nu)L_{n-1}(\nu) = 1. \quad (3.6)$$

This is the equation to determine the characteristic exponent  $\nu$ . By using  $\alpha_{-n}^{-\nu-1} = \gamma_n^\nu$  and  $\beta_{-n}^{-\nu-1} = \beta_n^\nu$ , we can prove that if  $\nu$  is a solution of the transcendental equation, then  $-\nu - 1$  is a solution, similarly to the proof given in Ref.[1] for the Kerr geometry case.

We can also prove that

$$a_{-n}^{-\nu-1} = a_n^\nu, \quad (3.7)$$

by using the recurrence relation and by taking the initial condition  $a_0^\nu = a_0^{-\nu-1}$ .

*(a-3) The solution which satisfies the outgoing boundary condition at the outer horizon*



This solution is simply given by

$$R_{out;\{0,1\};s}^\nu = (\Delta_r^{-s} R_{in;\{0,1\};-s}^\nu)^* , \quad (3.8)$$

which is proved in Appendix. The convergence region of this solution is the same as that for  $R_{in;\{0,1\};s}^\nu$ .

(a-4) *Solutions specified by characteristic exponents*

By using the relation in Eq.(3.7), we can show that  $R_{in;\{0,1\};s}^{-\nu-1}(x) = R_{in;\{0,1\};s}^\nu(x)$ . This means that  $R_{in;\{0,1\};s}^\nu(x)$  is expressed by a sum of two independent solutions with characteristic exponents,  $\nu$  and  $-\nu - 1$ . We can explicitly show with  $z = 1 - x$  that

$$R_{in;\{0,1\};s}^\nu(x) = R_{\{0,1\};s}^\nu(z) + R_{\{0,1\};s}^{-\nu-1}(z) , \quad (3.9)$$

where

$$\begin{aligned} R_{\{0,1\};s}^\nu(z) &= z^{A_2}(z-1)^{A_1} \left(1 - \frac{z}{z_r}\right)^{A_3} \left(1 - \frac{z}{z_\infty}\right)^{2s+1} \Gamma(\gamma) \\ &\times \sum_{n=-\infty}^{\infty} a_n^\nu \frac{\Gamma(2n+2\nu+1)}{\Gamma(n+\nu+\frac{\omega}{2}+\frac{1}{2})\Gamma(n+\nu+\gamma-\frac{\omega}{2}+\frac{1}{2})} z^{n+\nu-\frac{\omega}{2}+\frac{1}{2}} \\ &\times F\left(-n-\nu+\frac{\omega}{2}-\frac{1}{2}, -n-\nu+\gamma-\frac{\omega}{2}-\frac{1}{2}; -2n-2\nu; \frac{1}{z}\right) . \end{aligned} \quad (3.10)$$

(a-5) *The convergence region of the series for  $R_{in;\{0,1\};s}^\nu(x)$ , etc.*

Firstly, we consider the convergence region for  $R_{in;\{0,1\};s}^\nu(x)$ . We find that

$$\lim_{n \rightarrow \infty} \left| \frac{F_{n+1}^\nu}{F_n^\nu} \right| = \lim_{n \rightarrow -\infty} \left| \frac{F_n^\nu}{F_{n+1}^\nu} \right| = \max\{|e^\xi|, |e^{-\xi}|\} \geq 1 , \quad (3.11)$$

where we used the abbreviated expression  $F_n^\nu$  for the hypergeometric function appeared in Eq.(3.1) as

$$F_n^\nu \equiv F\left(-n-\nu+\frac{\omega}{2}-\frac{1}{2}, n+\nu+\frac{\omega}{2}+\frac{1}{2}; \gamma; x\right) , \quad (3.12)$$

and

$$e^{\pm\xi} = 1 - 2x \pm \sqrt{(1-2x)^2 - 1} . \quad (3.13)$$

On the other hand, the ratio of coefficients converge either  $e^{\xi_r} > 1$  or  $e^{-\xi_r} < 1$ . Taking account for Eq.(3.11), we have to require their limits as follows;

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}^\nu}{a_n^\nu} = \lim_{n \rightarrow -\infty} \frac{a_n^\nu}{a_{n+1}^\nu} = e^{-\xi_r} , \quad (3.14)$$

where

$$e^{\xi_r} = 1 - 2x_r + \sqrt{(1 - 2x_r)^2 - 1} > 1 \quad (x_r < 0) . \quad (3.15)$$

In order for the series of hypergeometric functions converges, it is required that

$$\max\{|e^\xi|, |e^{-\xi}|\} < e^{\xi_r} . \quad (3.16)$$

Now we parametrize  $x$  as  $1 - 2x = (t + \frac{1}{t})/2$  where  $|t| \geq 1$ . This transformation maps a circle in the complex  $t$  plane,  $|t| = C$  ( $C$ :constant) to an ellipse in the complex  $x$  plane. The condition of the convergence in Eq.(3.16) is expressed by  $|t| < e^{\xi_r}$ , i.e., inside the circle. In the complex  $x$  plane, the convergence region is inside the ellipse with foci 0 and 1, where the major axis is along the real axis extending from  $x_r$  to  $1 - x_r$ .

The convergence region for  $R_{out;\{0,1\};s}^\nu(x)$  is the same as  $R_{in;\{0,1\};s}^\nu(x)$  as seen from Eq.(3.8). The convergence region for  $R_{\{0,1\};s}^\nu(z)$  is also the same as  $R_{in;\{0,1\};s}^\nu(x)$ , because  $R_{\{0,1\};s}^\nu(z)$  and  $R_{\{0,1\};s}^{-\nu-1}(z)$  are expressed by linear combinations of  $R_{in;\{0,1\};s}^\nu$  and  $R_{out;\{0,1\};s}^\nu$ .

In summary, the convergence region of series of all solutions is inside of the ellipse with foci 0 and 1, where the major axis is along the real axis extending from  $x_r$  to  $1 - x_r$ . If we confine  $x$  to the physical region, the convergence region is  $x > x_r$ , i.e.,  $r < r'_+$ , with  $r'_+$  being the de Sitter horizon.

## (B) Solutions convergent around the de Sitter horizon

Here, we use the variable  $z$  rather than  $x$  and construct solutions which converge around the de Sitter horizon.

### (b-1) Solutions which are specified by the characteristic exponents

Firstly, we consider the solution with the characteristic exponent  $\nu$ . For this, we factor out the singularity at  $z = z_\infty$  by the transformation

$$R_{\{z_r, \infty\};s}^\nu(z) = z^{A_2} (z - 1)^{A_1} \left(1 - \frac{z}{z_r}\right)^{A_3} \left(1 - \frac{z}{z_\infty}\right)^{2s+1} g_{\{z_r, \infty\};s}^\nu(z), \quad (3.17)$$

where  $A_i$  ( $i = 1, 2, 3$ ) takes either  $A_{i+}$  or  $A_{i-}$ . Then,  $g_{\{z_r, \infty\};s}^\nu(z)$  should satisfy the following Heun's equation

$$\left\{ \frac{d^2}{dz^2} + \left[ \frac{\delta}{z} + \frac{\gamma}{z-1} + \frac{\epsilon}{z-z_r} \right] \frac{d}{dz} + \frac{\sigma_+ \sigma_- z - v - \sigma_+ \sigma_-}{z(z-1)(z-z_r)} \right\} g_{\{z_r, \infty\};s}^\nu(z) = 0. \quad (3.18)$$

Now we make the variable change  $\zeta = z_r/z$  and the transformation  $g_{\{z_r, \infty\};s}^\nu(z) = \zeta^{\sigma_+} h_{1;s}^\nu(\zeta)$ , we find a Heun's equation

$$\left\{ \frac{d^2}{d\zeta^2} + \left[ \frac{\sigma_+ - \sigma_- + 1}{\zeta} + \frac{\epsilon}{\zeta - 1} + \frac{\gamma}{\zeta - z_r} \right] \frac{d}{d\zeta} \right.$$

$$\left. + \frac{\sigma_+(\sigma_+ + \gamma - \omega)\zeta - v - z_r\sigma_+(\sigma_+ - \omega) - \sigma_+\gamma}{\zeta(\zeta - 1)(\zeta - z_r)} \right\} h_{1;s}^\nu(\zeta) = 0, \quad (3.19)$$

where we used the Heun's constraint in Eq.(2.17). Then, we find a solution

$$\begin{aligned} g_{1;\{z_r, \infty\};s}^\nu &= \frac{\Gamma(\nu + \sigma_- - \frac{\omega}{2} + \frac{1}{2})}{\Gamma(\nu - \sigma_- + \frac{\omega}{2} + \frac{3}{2})} \left(\frac{z_r}{z}\right)^{\sigma_+} \sum_{n=-\infty}^{\infty} a_n^\nu(-)^n \frac{\Gamma(n + \nu - \sigma_- + \frac{\omega}{2} + \frac{3}{2})}{\Gamma(n + \nu + \sigma_- - \frac{\omega}{2} + \frac{1}{2})} \\ &\times F\left(-n - \nu + \sigma_+ - \frac{\omega}{2} - \frac{1}{2}, n + \nu + \sigma_+ - \frac{\omega}{2} + \frac{1}{2}; \sigma_+ - \sigma_- + 1; \frac{z_r}{z}\right), \end{aligned} \quad (3.20)$$

and the other by exchanging  $\sigma_+$  with  $\sigma_-$

$$g_{2;\{z_r, \infty\};s}^\nu = g_{1;\{z_r, \infty\};s}^\nu \Big|_{\sigma_+ \longleftrightarrow \sigma_-}. \quad (3.21)$$

They satisfy the relations  $g_{i;\{z_r, \infty\};s}^{-\nu-1} = g_{i;\{z_r, \infty\};s}^\nu$  ( $i=1, 2$ ), so that a combination of them forms a solution with the characteristic exponent  $\nu$ . In fact, we find

$$\begin{aligned} g_{\{z_r, \infty\};s}^\nu &= \frac{(-)^{-\nu + \frac{\omega}{2} - \frac{1}{2}}}{\Gamma(\sigma_+ - \epsilon + 1)} \left[ (-)^{-\sigma_+} \frac{\Gamma(\sigma_- - \sigma_+)\Gamma(\nu - \sigma_- + \frac{\omega}{2} + \frac{3}{2})}{\Gamma(\nu + \sigma_- - \frac{\omega}{2} + \frac{1}{2})} g_{1;\{z_r, \infty\};s}^\nu \right. \\ &\quad \left. + (-)^{-\sigma_-} \frac{\Gamma(\sigma_+ - \sigma_-)\Gamma(\nu - \sigma_+ + \frac{\omega}{2} + \frac{3}{2})}{\Gamma(\nu + \sigma_+ - \frac{\omega}{2} + \frac{1}{2})} g_{2;\{z_r, \infty\};s}^\nu \right]. \end{aligned} \quad (3.22)$$

gives the solution. By substituting this, we find

$$\begin{aligned} R_{\{z_r, \infty\};s}^\nu(z) &= z^{A_2}(z-1)^{A_1} \left(1 - \frac{z}{z_r}\right)^{A_3} \left(1 - \frac{z}{z_\infty}\right)^{2s+1} \frac{1}{\Gamma(\sigma_+ - \epsilon + 1)} \\ &\times \sum_{n=-\infty}^{\infty} a_n^\nu \frac{\Gamma(n + \nu - \sigma_+ + \frac{\omega}{2} + \frac{3}{2})\Gamma(n + \nu - \sigma_- + \frac{\omega}{2} + \frac{3}{2})}{\Gamma(2n + 2\nu + 2)} \left(\frac{z}{z_r}\right)^{n + \nu - \frac{\omega}{2} + \frac{1}{2}} \\ &\times F\left(n + \nu + \sigma_+ - \frac{\omega}{2} + \frac{1}{2}, n + \nu + \sigma_- - \frac{\omega}{2} + \frac{1}{2}; 2n + 2\nu + 2; \frac{z}{z_r}\right). \end{aligned} \quad (3.23)$$

is a solution specified by the characteristic exponent  $\nu$ . Another independent solution with the characteristic exponent  $-\nu - 1$  is given by

$$R_{\{z_r, \infty\};s}^{-\nu-1}(z). \quad (3.24)$$

*(b-2) Solutions which satisfy the incoming and the outgoing boundary conditions at the de Sitter horizon*

The solution  $R_{\{z_r, \infty\};s}^\nu(z)$  is expressed by the sum of solutions which satisfy the incoming and the outgoing boundary conditions at the de Sitter horizon. We find

$$R_{\{z_r, \infty\};s}^\nu(z) = R_{\{z_r, \infty\};s}^{1;\nu}(z) + R_{\{z_r, \infty\};s}^{2;\nu}(z), \quad (3.25)$$

where

$$\begin{aligned}
R_{\{z_r, \infty\};s}^{1;\nu}(z) &= z^{A_2}(z-1)^{A_1} \left(1 - \frac{z}{z_r}\right)^{A_3-\epsilon+1} \left(1 - \frac{z}{z_\infty}\right)^{2s+1} \frac{\Gamma(\epsilon-1)}{\Gamma(\sigma_+ - \epsilon + 1)} \\
&\times \sum_{n=-\infty}^{\infty} a_n^\nu \frac{\Gamma(n+\nu-\sigma_+ + \frac{\omega}{2} + \frac{3}{2})\Gamma(n+\nu-\sigma_- + \frac{\omega}{2} + \frac{3}{2})}{\Gamma(n+\nu+\sigma_+ - \frac{\omega}{2} + \frac{1}{2})\Gamma(n+\nu+\sigma_- - \frac{\omega}{2} + \frac{1}{2})} \left(\frac{z}{z_r}\right)^{n+\nu-\frac{\omega}{2}+\frac{1}{2}} \\
&\times F\left(n+\nu-\sigma_+ + \frac{\omega}{2} + \frac{3}{2}, n+\nu-\sigma_- + \frac{\omega}{2} + \frac{3}{2}; 2-\epsilon; 1 - \frac{z}{z_r}\right),
\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}
R_{\{z_r, \infty\};s}^{2;\nu}(z) &= z^{A_2}(z-1)^{A_1} \left(1 - \frac{z}{z_r}\right)^{A_3} \left(1 - \frac{z}{z_\infty}\right)^{2s+1} \frac{\Gamma(1-\epsilon)}{\Gamma(\sigma_+ - \epsilon + 1)} \sum_{n=-\infty}^{\infty} a_n^\nu \\
&\times \left(\frac{z}{z_r}\right)^{n+\nu-\frac{\omega}{2}+\frac{1}{2}} F\left(n+\nu+\sigma_+ - \frac{\omega}{2} + \frac{1}{2}, n+\nu+\sigma_- - \frac{\omega}{2} + \frac{1}{2}; \epsilon; 1 - \frac{z}{z_r}\right).
\end{aligned} \tag{3.27}$$

(b-3) The convergence region for  $R_{\{z_r, \infty\};s}^\nu(z)$ , etc.

Firstly, we consider  $g_{i;\{z_r, \infty\};s}^\nu$  ( $i = 1, 2$ ) in Eqs.(3.20) and (3.21). By comparing Eqs.(3.20) and (3.21) with (3.1), we find that the convergence region of the series for  $g_{i;\{z_r, \infty\};s}^\nu(z)$  ( $i = 1, 2$ ) in complex  $z_r/z$  plane is inside the ellipse with foci are 0 and 1, where with the major axis is along the real axis extending from  $x_r$  to  $1 - x_r$ , i.e.,  $x_r < z_r/z < 1 - x_r$ . Keeping in mind that  $z = 1 - x$  and  $z_r = 1 - x_r$ , we find that the convergence region is  $x < 0$ , if we confine  $x$  to physical values. This region corresponds to  $r > r_+$  with  $r_+$  being the outer horizon.

Now we observe that  $R_{\{z_r, \infty\};s}^\nu(z)$  and  $R_{\{z_r, \infty\};s}^{-\nu-1}(z)$  are expressed by linear combination of  $g_{i;\{z_r, \infty\};s}^\nu(z)$  ( $i = 1, 2$ ), and also  $R_{\{z_r, \infty\};s}^{1;\nu}(z)$  and  $R_{\{z_r, \infty\};s}^{2;\nu}(z)$  are expressed by linear combinations of  $R_{\{z_r, \infty\};s}^\nu(z)$  and  $R_{\{z_r, \infty\};s}^{-\nu-1}(z)$ . Thus, we conclude that the convergence region of all these functions are the same region, i.e.,  $r > r_+$ .

(C) The matching of two solutions

We consider the matching of the solution which is convergent for  $r < r'_+$  and the solution which is convergent for  $r > r_+$ . We make the matching in the region  $r_+ < r < r'_+$  where both series are convergent. The matching can be made between two solutions with the same characteristic exponent. Explicitly, we require that

$$R_{\{0,1\};s}^\nu(z) = K_\nu R_{\{z_r, \infty\};s}^\nu(z). \tag{3.28}$$

for the region  $r_+ < r < r'_+$ . The proportionality constant  $K_\nu$  is determined by comparing the coefficients of  $z^{p+\nu-\frac{\omega}{2}+\frac{1}{2}}$ . The result is

$$K_\nu = \frac{z_r^{p+\nu-\frac{\omega}{2}+\frac{1}{2}} \Gamma(\gamma) \Gamma(\sigma_+ - \epsilon + 1)}{\Gamma(p + \nu - \frac{\omega}{2} + \frac{3}{2}) \Gamma(p + \nu + \delta - \frac{\omega}{2} + \frac{1}{2}) \Gamma(p + \nu + \sigma_+ - \frac{\omega}{2} + \frac{1}{2}) \Gamma(p + \nu + \sigma_- - \frac{\omega}{2} + \frac{1}{2})} \\ \times \left[ \sum_{n=p}^{\infty} a_n^\nu \frac{(-)^{n-p} \Gamma(n + \nu - \frac{\omega}{2} + \frac{3}{2}) \Gamma(n + \nu + \delta - \frac{\omega}{2} + \frac{1}{2}) \Gamma(p + n + 2\nu + 1)}{\Gamma(n + \nu + \frac{\omega}{2} + \frac{1}{2}) \Gamma(n + \nu + \gamma - \frac{\omega}{2} + \frac{1}{2}) (n-p)!} \right] \\ \times \left[ \sum_{n=-\infty}^p a_n^\nu \frac{\Gamma(n + \nu - \sigma_+ + \frac{\omega}{2} + \frac{3}{2}) \Gamma(n + \nu - \sigma_- + \frac{\omega}{2} + \frac{3}{2})}{\Gamma(n + \nu + \sigma_+ - \frac{\omega}{2} + \frac{1}{2}) \Gamma(n + \nu + \sigma_- - \frac{\omega}{2} + \frac{1}{2}) \Gamma(p + n + 2\nu + 2) (p-n)!} \right]^{-1}. \quad (3.29)$$

The constant  $K_\nu$  includes an integer  $p$ , but is independent of it.

(D) Solutions valid in the entire region

The solution which satisfies the incoming boundary condition at the outer horizon of black hole and convergent in the entire region is expressed by

$$R_{in;s}^\nu = \tilde{A}_s R_{in;\{0,1\};s}^\nu \\ = \tilde{A}_s \left[ K_\nu(s) R_{\{z_r, \infty\};s}^\nu(z) + K_{-\nu-1}(s) R_{\{z_r, \infty\};s}^{-\nu-1}(z) \right], \quad (3.30)$$

where  $\tilde{A}_s$  is the normalization constant. The first can be used for  $r < r'_+$  with  $r'_+$  being the de Sitter horizon and the second can be used for  $r > r_+$  with  $r_+$  being the outer horizon.

The other solution which satisfies the other boundary condition is obtained from a linear combination of two independent solutions,  $R_{in;s}^\nu$  and

$$R_{out;s}^\nu = (\Delta_r^{-s} R_{in;-s}^\nu)^*. \quad (3.31)$$

which satisfies the outgoing boundary condition at the outer horizon.

## 4 The Teukolsky-Starobinsky identities

We discuss the Teukolsky-Starobinsky (T-S) identities for perturbed fields in Kerr-Newman-de Sitter geometries, based on Teukolsky equation defined in Eq.(2.3), although this equation is not applied for electromagnetic field and gravitational field because they couple each other in this geometry. Nevertheless, we make the analysis because we want to see the mathematical structure of Teukolsky equation in detail and also we want to treat the T-S identities in the unified manner. Of course, the following analysis is valid for spin

1 fields which do not couple to electric charge and needless to say that it is valid for all fields for Kerr-de Sitter geometries. The T-S identities are expressed by

$$\begin{aligned}\Delta_r^s (\mathcal{D}_Q^\dagger)^{2s} \Delta_r^s R_s &= C_s^* R_{-s} \quad (\text{T-S identity (A)}), \\ (\mathcal{D}_Q)^{2s} R_{-s} &= C_s R_s \quad (\text{T-S identity (B)}),\end{aligned}\tag{4.1}$$

where  $C_s$  are Starobinsky constants and  $\mathcal{D}_Q$  and  $\mathcal{D}_Q^\dagger$  are defined by

$$\begin{aligned}\mathcal{D}_Q^\dagger &= \partial_r + i \frac{1+\alpha}{\Delta_r} \left( K - \frac{eQ}{1+\alpha} r \right), \\ \mathcal{D}_Q &= \partial_r - i \frac{1+\alpha}{\Delta_r} \left( K - \frac{eQ}{1+\alpha} r \right).\end{aligned}\tag{4.2}$$

The T-S identities are interesting because of their mathematical structure as well as to fix the relative normalization between the solutions specified by  $s$  and  $-s$ .

#### (A) Differential operators

Since the solutions are expressed in terms of  $x$  or  $z$ , we rewrite the differential operator  $\mathcal{D}_Q^\dagger$  as

$$\begin{aligned}\mathcal{D}_Q^\dagger &= -\frac{r_+ - r'_-}{(r_+ - r_-)(r_- - r'_-)} (-x)^{-a_1} (1-x)^{-a_2} \left( \frac{x-x_r}{1-x_r} \right)^{-a_3} \left( \frac{x-x_\infty}{1-x_\infty} \right)^2 \\ &\quad \times \frac{d}{dx} (-x)^{a_1} (1-x)^{a_2} \left( \frac{x-x_r}{1-x_r} \right)^{a_3} \\ &= \frac{r_+ - r'_-}{(r_+ - r_-)(r_- - r'_-)} z^{-a_2} (z-1)^{-a_1} \left( 1 - \frac{z}{z_r} \right)^{-a_3} \left( 1 - \frac{z}{z_\infty} \right)^2 \\ &\quad \times \frac{d}{dz} z^{a_2} (z-1)^{a_1} \left( 1 - \frac{z}{z_r} \right)^{a_3}.\end{aligned}\tag{4.3}$$

The  $\mathcal{D}_Q$  is derived by taking the complex conjugation of  $\mathcal{D}_Q^\dagger$ .

#### (B) Parametrizations of solutions

Solutions are specified by the choice of  $A_i$  in Eq.(2.11) so that we have to specify these parameters to fix them.

##### (b-1) A specific choice of $A_i$

Here we consider the solution which satisfies the incoming boundary condition at the outer horizon and thus we choose  $A_1 = A_{1-}$ . For others, we choose  $A_2 = A_{2+}$  and  $A_3 = A_{3-}$ . Then we have

$$A_1 = -a_1 - s, \quad A_2 = a_2, \quad A_3 = -a_3 - s, \tag{4.4}$$

so that we find

$$\begin{aligned}
\sigma_+ &= 2a_2 - s + 1, \quad \sigma_- = -2a_1 - 2a_3 + 1, \\
\gamma &= -2a_1 - s + 1, \quad \delta = 2a_2 + s + 1, \quad \epsilon = -2a_3 - s + 1, \\
\omega &\equiv \gamma + \delta - 1 = \sigma_+ + \sigma_- - \epsilon = -2a_1 + 2a_2 + 1.
\end{aligned} \tag{4.5}$$

Then, from Eqs.(3.1) and (3.30) the solution is given by the following two expressions which have different regions of convergence,

$$\begin{aligned}
R_{in;s}^\nu &= \tilde{A}_s(-x)^{-s-a_1}(1-x)^{a_2} \left( \frac{x-x_r}{1-x_r} \right)^{-s-a_3} \left( \frac{x-x_\infty}{1-x_\infty} \right)^{2s+1} \\
&\times \sum_{n=-\infty}^{\infty} a_n^\nu(s) F(-n-\nu-a_1+a_2, n+\nu-a_1+a_2+1; -2a_1-s+1; x),
\end{aligned} \tag{4.6}$$

which is convergent for  $r < r'_+$  (the expression (A)) and

$$R_{in;s}^\nu = \tilde{A}_s \left[ K_\nu(s) R_{\{z_r, \infty\};s}^\nu(z) + K_{-\nu-1}(s) R_{\{z_r, \infty\};s}^{-\nu-1}(z) \right], \tag{4.7}$$

where

$$\begin{aligned}
R_{\{z_r, \infty\};s}^\nu(z) &= z^{a_2}(z-1)^{-s-a_1} \left( 1 - \frac{z}{z_r} \right)^{-s-a_3} \left( 1 - \frac{z}{z_\infty} \right)^{2s+1} \\
&\times \sum_{n=-\infty}^{\infty} a_n^\nu(s) \frac{\Gamma(n+\nu+a_3-a_4+1)\Gamma(n+\nu-a_1-a_2+s+1)}{\Gamma(2a_2+2a_3+1)\Gamma(2n+2\nu+2)} \left( \frac{z}{z_r} \right)^{n+\nu+a_1-a_2} \\
&\times F\left(n+\nu+a_1+a_2-s+1, n+\nu-a_3+a_4+1; 2n+2\nu+2; \frac{z}{z_r}\right),
\end{aligned} \tag{4.8}$$

which is convergent for  $r > r_+$  (the expression (B)).

(b-2) Another choice of  $A_i$

It may be interesting to ask the relation between solutions with different choices of  $A_i$ . To answer this question, we consider a generic Heun's equation,

$$\left\{ \frac{d^2}{dz^2} + \left[ \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a_H} \right] \frac{d}{dz} + \frac{\sigma_+\sigma_-z-q}{z(z-1)(z-a_H)} \right\} f(z) = 0, \tag{4.9}$$

with  $\sigma_+ + \sigma_- - \epsilon = \gamma + \delta - 1$ . Here we express a particular solution as  $Hf(a_H, q; \sigma_+, \sigma_-, \gamma, \delta; z)$  which is regular at  $z = 0$  and is normalized to be one at  $z = 0$ . By setting

$$f(z) = \left( 1 - \frac{z}{a_H} \right)^{1-\epsilon} g(z), \tag{4.10}$$

the above Heun's equation is transformed to another form as

$$\left\{ \frac{d^2}{dz^2} + \left[ \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{2-\epsilon}{z-a_H} \right] \frac{d}{dz} + \frac{(\gamma+\delta-\sigma_+)(\gamma+\delta-\sigma_-)z-q-\gamma(1-\epsilon)}{z(z-1)(z-a_H)} \right\} g(z) = 0, \quad (4.11)$$

where we used  $1-\epsilon = \gamma + \delta - \sigma_+ - \sigma_-$ . The solution which is normalized and regular at  $z = 0$  is given by  $Hf(a_H, q + \gamma(1-\epsilon); \gamma + \delta - \sigma_+, \gamma + \delta - \sigma_-, \gamma, \delta; z)$ . Now we have two different expressions of the solution of Eq.(4.9). Both are regular at  $z = 0$  and are normalized to be one at  $z = 0$ , so that two solutions should be equal each other,

$$\begin{aligned} & Hf(a, q; \sigma_+, \sigma_-, \gamma, \delta; z) \\ &= \left(1 - \frac{z}{a_H}\right)^{1-\epsilon} Hf(a_H, q + \gamma(1-\epsilon); \gamma + \delta - \sigma_+, \gamma + \delta - \sigma_-, \gamma, \delta; z). \end{aligned} \quad (4.12)$$

We apply this fact to the incoming solution at outer horizon in Eq.(4.6), which is expressed by

$$\begin{aligned} R_{in;s}^\nu &= \left( \tilde{A}_s \sum_{n=-\infty}^{\infty} a_n^\nu(s) \right) (-x)^{-s-a_1} (1-x)^{a_2} \left( \frac{x-x_r}{1-x_r} \right)^{-s-a_3} \left( \frac{x-x_\infty}{1-x_\infty} \right)^{2s+1} \\ &\times Hf(x_r, -v; \sigma_+, \sigma_-, \gamma, \delta; x), \end{aligned} \quad (4.13)$$

where

$$\begin{aligned} & Hf(x_r, -v; \sigma_+, \sigma_-, \gamma, \delta; x) = \\ & \left[ \sum_{n=-\infty}^{\infty} a_n^\nu(s) \right]^{-1} \sum_{n=-\infty}^{\infty} a_n^\nu(s) F\left(-n-\nu + \frac{\omega}{2} - \frac{1}{2}, n+\nu + \frac{\omega}{2} + \frac{1}{2}; \gamma; x\right), \end{aligned} \quad (4.14)$$

By using the relation between Heun's functions in Eq.(4.12), this solution is at the same time expressed by

$$\begin{aligned} R_{in;s}^\nu &= \tilde{A}_s \left( \frac{-x_r}{1-x_r} \right)^{-s-2a_3} \left[ \sum_{n=-\infty}^{\infty} a_n^\nu(s) \right] (-x)^{-s-a_1} (1-x)^{a_2} \left( \frac{x-x_r}{1-x_r} \right)^{a_3} \left( \frac{x-x_\infty}{1-x_\infty} \right)^{2s+1} \\ &\times Hf(a_H, -v' + \gamma(1-\epsilon); \gamma + \delta - \sigma_+, \gamma + \delta - \sigma_-, \gamma, \delta; x), \end{aligned} \quad (4.15)$$

where

$$\begin{aligned} & Hf(a_H, -v' + \gamma(1-\epsilon); \gamma + \delta - \sigma_+, \gamma + \delta - \sigma_-, \gamma, \delta; x) \\ &= \left[ \sum_{n=-\infty}^{\infty} b_n^\nu(s) \right]^{-1} \sum_{n=-\infty}^{\infty} b_n^\nu(s) F\left(-n-\nu + \frac{\omega}{2} - \frac{1}{2}, n+\nu + \frac{\omega}{2} + \frac{1}{2}; \gamma; x\right). \end{aligned} \quad (4.16)$$



Here the coefficients are defined by

$$b_n^\nu(s) = a_n^\nu(s) \frac{\Gamma(\nu + \sigma_+ - \frac{\omega}{2} + \frac{1}{2})\Gamma(\nu + \sigma_- - \frac{\omega}{2} + \frac{1}{2})}{\Gamma(\nu + \frac{\omega}{2} - \sigma_+ + \frac{3}{2})\Gamma(\nu + \frac{\omega}{2} - \sigma_- + \frac{3}{2})} \times \frac{\Gamma(n + \nu + \frac{\omega}{2} - \sigma_+ + \frac{3}{2})\Gamma(n + \nu + \frac{\omega}{2} - \sigma_- + \frac{3}{2})}{\Gamma(n + \nu + \sigma_+ - \frac{\omega}{2} + \frac{1}{2})\Gamma(n + \nu + \sigma_- - \frac{\omega}{2} + \frac{1}{2})} . \quad (4.17)$$

The difference of coefficients are due to the change of parameters in the recurrence relation which coefficients satisfy. This solution in Eq.(4.16) is the one which corresponds to  $A_1 = A_{1-}$ ,  $A_2 = A_{2+}$  and  $A_3 = A_{3+}$ .

### (C) Various relations

The characteristic exponent  $\nu$ , coefficients  $a_n^\nu$ ,  $b_n^\nu$  and the proportionality constant  $K_\nu$  satisfy the following relations, which are important to show that the solutions satisfy the T-S identities.

#### (c-1) The relation between $\nu(s)$ and $\nu(-s)$

As we prove in Appendix, we have

$$\nu(s) = \nu(-s) , \quad \nu(s) = \nu(s)^* . \quad (4.18)$$

These relation is vital for the T-S identities, because the T-S identities are differential transformation from the solution specified by  $s$  to that by  $-s$ .

#### (c-2) The relation between coefficients and $K_\nu$

By examining the recurrence relation in Eq.(3.2), we find

$$a_n^\nu(-s) = \left| \frac{\Gamma(\nu + a_1 + a_2 - s + 1)}{\Gamma(\nu + a_1 + a_2 + s + 1)} \right|^2 \left| \frac{\Gamma(n + \nu + a_1 + a_2 + s + 1)}{\Gamma(n + \nu + a_1 + a_2 - s + 1)} \right|^2 a_n^\nu(s) , \quad (4.19)$$

where we chose  $a_0^\nu(-s) = a_0^\nu(s)$ . Also, by using this relation, we find

$$b_n^\nu(-s) = b_n^\nu(s) . \quad (4.20)$$

and

$$\frac{K_\nu(s)}{K_\nu(-s)} = \frac{K_{-\nu-1}(s)}{K_{-\nu-1}(-s)} = \frac{\Gamma(-2a_1 - s + 1)}{\Gamma(-2a_1 + s + 1)} . \quad (4.21)$$

#### (c-3) Useful mathematical formula

The following mathematical relations are needed:

$$\left[ \left( \frac{x - x_\infty}{1 - x_\infty} \right)^2 \frac{d}{dx} \right]^k = \left( \frac{x - x_\infty}{1 - x_\infty} \right)^{k+1} \left( \frac{d}{dx} \right)^k \left( \frac{x - x_\infty}{1 - x_\infty} \right)^{k-1} ,$$

$$\begin{aligned}
& \left( \frac{d}{dx} \right)^k (1-x)^{A+B-C} F(A, B; C; x) \\
&= \frac{\Gamma(C-A+k)\Gamma(C-B+k)\Gamma(C)}{\Gamma(C-A)\Gamma(C-B)\Gamma(C+k)} (1-x)^{A+B-C-k} F(A, B; C+k; x) , \\
& \left( \frac{d}{dz} \right)^k z^{A+k-1} F(A, B; C; z) = \frac{\Gamma(A+k)}{\Gamma(A)} z^{A-1} F(A+k, B; C; z) .
\end{aligned} \tag{4.22}$$

(D) Teukolsky-Starobinsky identity

First we discuss the solution given by the expression (A) in Eq.(4.6) and show analytically that this solution satisfies the T-S identity (A) when we take the relative normalization between  $R_{in;s}^\nu$  and  $R_{in;-s}^\nu$  properly. In other words, we can fix the normalization factor for  $R_{in;s}^\nu$ ,  $\tilde{A}_s$  for  $s > 0$  by the T-S identity once we fix  $\tilde{A}_{-s} = 1$ . First we apply the operator  $\Delta_r^s (\mathcal{D}_Q^\dagger)^{2s} \Delta_r^s$  to the expression (A):

$$\begin{aligned}
& \Delta_r^s (\mathcal{D}_Q^\dagger)^{2s} \Delta_r^s R_{in;s}^\nu \\
&= \tilde{A}_s \left[ -\frac{\alpha}{a^2} (r_+ - r_-) (r'_+ - r_-) (r_- - r'_-) \right]^{2s} (-x)^{s-a_1} (1-x)^{s-a_2} \left( \frac{x-x_r}{1-x_r} \right)^{s-a_3} \\
& \quad \times \left( \frac{x-x_\infty}{1-x_\infty} \right)^{-4s} \left[ \left( \frac{x-x_\infty}{1-x_\infty} \right)^2 \frac{d}{dx} \right]^{2s} \left( \frac{x-x_\infty}{1-x_\infty} \right)^{-2s+1} (1-x)^{s+2a_2} \\
& \quad \times \sum_{n=-\infty}^{\infty} a_n^\nu(s) F(-n-\nu-a_1+a_2, n+\nu-a_1+a_2+1; -2a_1-s+1; x) \\
&= \tilde{A}_s \left[ -\frac{\alpha}{a^2} (r_+ - r_-) (r'_+ - r_-) (r_- - r'_-) \right]^{2s} \\
& \quad \times (-x)^{s-a_1} (1-x)^{s-a_2} \left( \frac{x-x_r}{1-x_r} \right)^{s-a_3} \left( \frac{x-x_\infty}{1-x_\infty} \right)^{-2s+1} \left( \frac{d}{dx} \right)^{2s} (1-x)^{s+2a_2} \\
& \quad \times \sum_{n=-\infty}^{\infty} a_n^\nu(s) F(-n-\nu-a_1+a_2, n+\nu-a_1+a_2+1; -2a_1-s+1; x) \\
&= \tilde{A}_s \left[ \frac{\alpha}{a^2} (r_+ - r_-) (r'_+ - r_-) (r_- - r'_-) \right]^{2s} \\
& \quad \times \frac{\Gamma(-2a_1-s+1)}{\Gamma(-2a_1+s+1)} \left| \frac{\Gamma(\nu+a_1+a_2+s+1)}{\Gamma(\nu+a_1+a_2-s+1)} \right|^2 R_{in;-s}^\nu,
\end{aligned} \tag{4.23}$$

where we used relations in Eqs.(4.22) and (4.19). Thus  $\tilde{A}_s$  is fixed by

$$\begin{aligned}
\tilde{A}_s &= C_s^* \left[ -\frac{a^2}{\alpha(r_+ - r_-)(r'_+ - r_-)(r'_- - r_-)} \right]^{2s} \\
& \quad \times \frac{\Gamma(-2a_1+s+1)}{\Gamma(-2a_1-s+1)} \left| \frac{\Gamma(\nu+a_1+a_2-s+1)}{\Gamma(\nu+a_1+a_2+s+1)} \right|^2 (s > 0) ,
\end{aligned} \tag{4.24}$$

provided  $\tilde{A}_{-s} = 1$  ( $s > 0$ ).

Next we show that the expression (B) of the solution in Eq.(4.7) satisfies the T-S identity (A). To the end we observe for  $s > 0$

$$\begin{aligned}
& \Delta_r^s (\mathcal{D}_Q^\dagger)^{2s} \Delta_r^s R_{\{z_r, \infty\}; s}^\nu \\
&= \left[ \frac{\alpha}{a^2} (r_+ - r_-) (r'_+ - r_-) (r_- - r'_-) \right]^{2s} z^{s-a_2} (z-1)^{s-a_1} \left( 1 - \frac{z}{z_r} \right)^{s-a_3} \\
&\quad \times \left( 1 - \frac{z}{z_\infty} \right)^{-4s} \left[ \left( 1 - \frac{z}{z_\infty} \right)^2 \frac{d}{dz} \right]^{2s} \left( 1 - \frac{z}{z_\infty} \right)^{-2s+1} z^{s+2a_2} \\
&\quad \times \sum_{n=-\infty}^{\infty} a_n^\nu(s) \frac{\Gamma(n+\nu+a_3-a_4+1) \Gamma(n+\nu-a_1-a_2+s+1)}{\Gamma(2a_2+2a_3+1) \Gamma(2n+2\nu+2)} \\
&\quad \times \left( \frac{z}{z_r} \right)^{n+\nu+a_1-a_2} F \left( n+\nu+a_1+a_2-s+1, n+\nu-a_3+a_4+1; 2n+2\nu+2; \frac{z}{z_r} \right) \\
&= \left[ \frac{\alpha}{a^2} (r_+ - r_-) (r'_+ - r_-) (r_- - r'_-) \right]^{2s} \\
&\quad \times z^{s-a_2} (z-1)^{s-a_1} \left( 1 - \frac{z}{z_r} \right)^{s-a_3} \left( 1 - \frac{z}{z_\infty} \right)^{-2s+1} \\
&\quad \times \left( \frac{d}{dz} \right)^{2s} z^{s+2a_2} \sum_{n=-\infty}^{\infty} a_n^\nu(s) \frac{\Gamma(n+\nu+a_3-a_4+1) \Gamma(n+\nu-a_1-a_2+s+1)}{\Gamma(2a_2+2a_3+1) \Gamma(2n+2\nu+2)} \\
&\quad \times \left( \frac{z}{z_r} \right)^{n+\nu+a_1-a_2} F \left( n+\nu+a_1+a_2-s+1, n+\nu-a_3+a_4+1; 2n+2\nu+2; \frac{z}{z_r} \right) \\
&= \left[ \frac{\alpha}{a^2} (r_+ - r_-) (r'_+ - r_-) (r_- - r'_-) \right]^{2s} \left| \frac{\Gamma(\nu+a_1+a_2+s+1)}{\Gamma(\nu+a_1+a_2-s+1)} \right|^2 R_{\{z_r, \infty\}; -s}^\nu,
\end{aligned} \tag{4.25}$$

where we used relations in Eqs.(4.22) and (4.19).

By exchanging  $\nu$  with  $-\nu-1$  and using the fact that  $2s$  is integer, we find from Eq.(4.25) that

$$\begin{aligned}
& \Delta_r^s (\mathcal{D}_Q^\dagger)^{2s} \Delta_r^s R_{\{z_r, \infty\}; s}^{-\nu-1} \\
&= \left[ \frac{\alpha}{a^2} (r_+ - r_-) (r'_+ - r_-) (r_- - r'_-) \right]^{2s} \left| \frac{\Gamma(\nu+a_1+a_2+s+1)}{\Gamma(\nu+a_1+a_2-s+1)} \right|^2 R_{\{z_r, \infty\}; -s}^{-\nu-1}.
\end{aligned} \tag{4.26}$$

Thus we find

$$\begin{aligned}
\Delta_r^s (\mathcal{D}_Q^\dagger)^{2s} \Delta_r^s R_{in; s}^\nu &= \tilde{A}_s \left[ \frac{\alpha}{a^2} (r_+ - r_-) (r'_+ - r_-) (r_- - r'_-) \right]^{2s} \\
&\quad \times \frac{\Gamma(-2a_1-s+1)}{\Gamma(-2a_1+s+1)} \left| \frac{\Gamma(\nu+a_1+a_2+s+1)}{\Gamma(\nu+a_1+a_2-s+1)} \right|^2
\end{aligned}$$

$$\begin{aligned}
& \times \left[ K_\nu(-s) R_{\{z_r, \infty\}; -s}^\nu(z) + K_{-\nu-1}(-s) R_{\{z_r, \infty\}; -s}^{-\nu-1}(z) \right], \\
& = C_s^* \left[ K_\nu(-s) R_{\{z_r, \infty\}; -s}^\nu(z) + K_{-\nu-1}(-s) R_{\{z_r, \infty\}; -s}^{-\nu-1}(z) \right], \\
& = C_s^* R_{in; -s}^\nu(z),
\end{aligned} \tag{4.27}$$

where we used the relation in Eqs.(4.21).

Next we examine the T-S identity (B) by using another parametrization of the solution given in Eq.(4.15). By applying the differential operator  $(\mathcal{D}_Q)^{2s}$  to this expression, we find

$$\begin{aligned}
(\mathcal{D}_0)^{2s} R_{in; -s}^\nu &= \left( \frac{-x_r}{1-x_r} \right)^{s-2a_3} \left[ \sum_{n=-\infty}^{\infty} a_n^\nu(-s) \right] \left[ \sum_{n=-\infty}^{\infty} b_n^\nu(-s) \right]^{-1} \left[ -\frac{r_+ - r'_-}{(r_+ - r_-)(r_- - r'_-)} \right]^{2s} \\
&\quad \times (-x)^{a_1} (1-x)^{a_2} \left( \frac{x-x_r}{1-x_r} \right)^{a_3} \left[ \left( \frac{x-x_\infty}{1-x_\infty} \right)^2 \frac{d}{dx} \right]^{2s} \left( \frac{x-x_\infty}{1-x_\infty} \right)^{-2s+1} (-x)^{s-2a_1} \\
&\quad \times \sum_{n=-\infty}^{\infty} b_n^\nu(-s) F(-n-\nu-a_1+a_2, n+\nu-a_1+a_2+1; -2a_1+s+1; x) \\
&= \frac{1}{\tilde{A}_s} \left( \frac{-x_r}{1-x_r} \right)^{2s} \left[ \sum_{n=-\infty}^{\infty} a_n^\nu(-s) \right] \left[ \sum_{n=-\infty}^{\infty} a_n^\nu(s) \right]^{-1} \left[ \frac{r_+ - r'_-}{(r_+ - r_-)(r_- - r'_-)} \right]^{2s} \\
&\quad \times \frac{\Gamma(-2a_1+s+1)}{\Gamma(-2a_1-s+1)} R_{in; s}^\nu,
\end{aligned} \tag{4.28}$$

where we used relations in Eqs.(4.22) and (4.20).

Thus the T-S identity implies that the following identity among the sums of coefficients should be satisfied,

$$\sum_{n=-\infty}^{\infty} a_n^\nu(-s) = |C_s|^2 \left[ \frac{a^2}{\alpha(r'_+ - r_+)(r_- - r'_-)} \right]^{2s} \left| \frac{\Gamma(\nu + a_1 + a_2 - s + 1)}{\Gamma(\nu + a_1 + a_2 + s + 1)} \right|^2 \sum_{n=-\infty}^{\infty} a_n^\nu(s). \tag{4.29}$$

## 5 Summary and discussions

In our previous paper [8], we showed that both angular and radial equations for the Teukolsky equation in Kerr-de Sitter (for all massless fields) and Kerr-Newman-de Sitter (massless fields except electromagnetic and gravitational fields) geometries are transformed to Heun's equation and derived analytic solutions in the form of series of hypergeometric functions. The solution of the radial equation presented in Ref.8 is convergent for  $r < r'_+$  with  $r'_+$  being the de Sitter horizon. In this paper, we constructed the analytic solution of the radial equation which is convergent for  $r_+ < r$  with  $r_+$  being the outer horizon. By matching these solutions in the region where both solutions converge, *i.e.*,  $r_+ < r < r'_+$ ,

we obtained the analytic solution valid for the entire region of  $r$ . We showed analytically that our solution satisfies the Teukolsky-Starobinsky identities which are quite nontrivial differential transformations.

Although the use of our solution in this paper is less clear than that for the Kerr geometry case, we believe that our solution will be relevant to the physical world, especially in the early stage of the universe where the cosmological constant plays important role. By using our solution, we can derive the analytic expression of the decay rate for emission of massless particles from the Kerr-de Sitter or the Kerr-Newman-de Sitter black hole [11], which can be used to examine a correspondence between quantum gravity on anti-de Sitter space and a conformal field theory defined on its boundary [12,13],

## Appendix

(a) The proof that  $\Delta_r^{-s} R_{-s}^*$  is a solution

Form the radial Teukolsky equation for  $R_s$ , we derive the equation for  $\Delta_r^s R_s$ . We find

$$\left\{ \Delta_r^s \frac{d}{dr} \Delta_r^{1-s} \frac{d}{dr} + \frac{1}{\Delta_r} \left[ (1+\alpha)^2 \left( K - \frac{eQr}{1+\alpha} \right)^2 - is(1+\alpha) \left( K - \frac{eQr}{1+\alpha} \right) \frac{d\Delta_r}{dr} \right] \right. \\ \left. + 4is(1+\alpha)\omega r - \frac{2\alpha}{a^2}(s-1)(2s-1)r^2 - s(1-\alpha) - 2iseQ - \lambda_s \right\} \Delta_r^s R_s(r) = 0 \quad (\text{A.1})$$

By using the relation  $\lambda_s = \lambda_{-s}$ , we find that this is the equation which  $(R_{-s})^*$  satisfies. By changing  $s$  to  $-s$ , we conclude that  $(\Delta_r^{-s} R_{-s})^*$  satisfies the same Teukolsky equation as  $R_s$  does.

(b) The proof of  $\nu(s) = \nu(-s)$

With this choice of  $A_i$  in Eq.(4.4), coefficients in the recurrence relations are expressed by

$$\begin{aligned} \alpha_n^\nu(s) &= -\frac{(n+\nu+a_1-a_2+1)(n+\nu+a_3-a_4+1)|n+\nu+a_1+a_2+s+1|^2}{2(n+\nu+1)(2n+2\nu+3)}, \\ \beta_n^\nu(s) &= \frac{(a_1-a_2)(a_3-a_4)[s^2+(a_1+a_2)(a_3+a_4)]}{2(n+\nu)(n+\nu+1)} + \left(\frac{1}{2} - x_r\right)(n+\nu)(n+\nu+1) \\ &\quad + \left[ \frac{1}{2} - \frac{(r_+ + r'_+)^2}{(r_+ - r_-)(r'_+ - r'_-)} \right] s^2 - a_1 a_3 + a_2 a_4 + \frac{1}{2} + x_r(a_1^2 + a_2^2) \\ &\quad - \frac{2a^4(1+\alpha)^2(r_- - r'_+)^2(r_- - r'_-)^2(r_+ - r'_-)(r'_+ - r'_-)}{\alpha^2 \mathcal{D} (r_+ - r_-)} \\ &\quad \times \left\{ -\omega^2 r_+^3(r_+ r_- - 2r_- r'_+ + r_+ r'_+) + 2a\omega(a\omega - m)r_+(r_- r'_+ - r_+^2) \right. \\ &\quad \left. - a^2(a\omega - m)^2(2r_+ - r_- - r'_+) + \frac{eQ}{1+\alpha}[\omega r_+^2(r_+ r_- + r_+^2 - 3r_- r'_+ + r_+ r'_+) \right. \\ &\quad \left. - a(a\omega - m)(r_+ r_- - 3r_+^2 + r_- r'_+ + r_+ r'_+)] + \left(\frac{eQ}{1+\alpha}\right)^2 r_+(-r_+^2 + r_- r'_+) \right\} \\ &\quad - \left[ \frac{2r_-'^2}{(r_+ - r_-)(r'_+ - r'_-)} + x_\infty \right] - \frac{a^2}{\alpha(r_+ - r_-)(r'_+ - r'_-)} \lambda_s, \\ \gamma_n^\nu(s) &= -\frac{(n+\nu-a_1+a_2)(n+\nu-a_3+a_4)|n+\nu+a_1+a_2-s|^2}{2(n+\nu)(2n+2\nu-1)}. \end{aligned} \quad (\text{A.2})$$

And we see that

$$\alpha_{n-1}^\nu(s) \gamma_n^\nu(s) = \frac{|n+\nu+a_1-a_2|^2 |n+\nu+a_3-a_4|^2 |(n+\nu+a_1+a_2)^2 - s^2|^2}{4(n+\nu)^2(2n+2\nu-1)(2n+2\nu+1)}, \quad (\text{A.3})$$

Now we observe that the transcendental equation which determine  $\nu$  given in Eq.(3.6) contains coefficients as  $\beta_n^\nu(s)$  and a special combination  $\alpha_{n-1}^\nu(s)\gamma_n^\nu(s)$ . From expressions in Eqs.(A.2) and (A.3), we find that both  $\beta_n^\nu(s)$  and  $\alpha_{n-1}^\nu(s)\gamma_n^\nu(s)$  are even functions of  $s$ . Furthermore, we see that the transcendental equation for  $\nu$  is an equation of real coefficients. Therefore if  $\nu(s)$  is a solution, then  $\nu(-s)$  is a solution, and also  $\nu(s)^*$  is a solution. By the uniqueness of the solution, we conclude that the solution  $\nu(s)$  is an even function and a real function of  $s$ .

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